Dirac-Born-Infeld/Tachyon Fields Equivalence And Tachyon Dynamics

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We investigate in detail the asymptotic properties of tachyon cosmology for a generic class of self-interaction potentials. The present approach relies on an existing (formal) mathematical equivalence between a Dirac-Born-Infeld (DBI) model – with a particular choice of the warp factor and of the potential for the DBI field – and standard tachyon cosmology, under an appropriate transformation of the DBI field. The above mathematical equivalence is used to generalize the dynamical systems study of tachyon cosmology to a wider class of self-interaction potentials beyond the (inverse) square-law and power-law ones. It is revealed that independent of the functional form of the potential, the matter-dominated solution and the ultra-relativistic (also matter-dominated) solution, are always associated with equilibrium points in the phase space of the tachyon models. The latter is always the past attractor, while the former is a saddle critical point. For inverse power-law potentials $V \propto \phi^{-2\lambda}$ the late-time attractor is always the de Sitter solution, while for sinh-like potentials $V \propto \sinh^{-\alpha}(\lambda\phi)$, depending on the region of parameter space, the late-time attractor can be either the inflationary tachyon-dominated solution or the matter-scaling (also inflationary) phase. In general, for most part of known quintessential potentials, the late-time dynamics will be associated either with de Sitter inflation, or with matter-scaling, or scalar field-dominated solutions.

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I. INTRODUCTION

Inflationary models of the universe have been studied from the string theory perspective because inflation provides an explanation for the homogeneity and isotropy of the early universe. Additionally, recent astrophysical observations indicate us that the universe is presently undergoing a phase of accelerated expansion that has been attributed to a peculiar kind of source of the Einstein's field equations acknowledged as "dark energy" [1].[1] The crucial feature of the dark energy which ensures an accelerated expansion of the universe is that it breaks the strong energy condition. The tachyon field arising in the context of string theory [4] provides one example of matter which does the job. The tachyon has been intensively studied during the last few years also in application to cosmology [5]-[14]; in this case one usually takes Sen's effective Lagrangian density for granted [4]:

$$\mathcal{L} = -V(\phi)\sqrt{1+(\nabla\phi)^2}, \tag{1}$$

and studies its cosmological consequences without worrying about the string-theoretical origin of the action itself. In the above equation ϕ is the scalar tachyon field, V-its self-interaction potential, and $(\nabla\phi)^2 \equiv g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi$.

Recently generalized nonlinear scalar field theories of the Dirac-Born-Infeld (DBI) type have been proposed [15]-[17]. As it is the case for Sen's tachyon field, these theories have attracted much attention in recent years due to their role in inflation based on string theory [18]. In the above scenarios the inflaton is identified with the position of a mobile D3-brane, moving on a compact 6-dimensional submanifold of spacetime (for reviews and references see [19]), which means that the inflaton is interpreted as an open string mode. The effects of such DBI-motivated fields in a Friedmann-Robertson-Walker (FRW) cosmology have been already studied by means of the dynamical systems tools, yielding scaling solutions when the equation of state of the perfect fluid is negative and in the ultra-relativistic limit [16]. Non-linear Born-Infeld scalar fields with negative potential have been also investigated within FRW cosmology [20]. The cosmological dynamics of a DBI field when the potential and the brane tension are arbitrary power-law or exponential functions of the DBI field, has been studied very recently in [21].

A dynamical systems study of the FRW cosmology

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within the simpler theories based on the tachyon Lagrangian (1) can be found in Ref.[14]. However the authors of [14] were able to study self-interaction potentials of the power-law type only. For more general potentials the corresponding system of ordinary differential equations in the phase space is not a closed system of equations any more, and one has to rely on the notion of "instantaneous critical points" whose physical relevance is unclear.

In the present paper we aim at studying the cosmological dynamics of the tachyon model given by (1), for a wider variety of tachyon self-interactions potentials than in Ref.[14]. To this end we shall exploit a formal mathematical equivalence existing between a DBI field – holding a particular relationship between the DBI potential and the warp factor – and the tachyon field, in combination with the application of an approach formerly used, for instance in Ref.[22], that allows to study a vast variety of self-interaction potentials. The above mathematical equivalence enables to extend the results of dynamical systems studies of DBI models – including results of studies already existing in the bibliography [14] – to tachyon field cosmology. We will be able, in particular, to note an equivalence between the results of Ref.[14] within the context of tachyon cosmology, and those in [21] for a DBI field with exponential potential and brane tension, not reported previously.

The paper has been organized in the following manner. The mathematical aspects of the particular DBI cosmological model considered, are given in section II. Section III is devoted to the study of the asymptotic properties of the above model for a generic variety of self-interaction potentials, through the application of the dynamical systems tools. In section IV we discuss the relevant aspects of the mathematical equivalence among the DBI model whose dynamics has been discussed in the previous section, and the corresponding tachyon model, so that the results of section III can be safely translated to the case of standard tachyon cosmology. It is demonstrated, in particular, that former studies within the standard tachyon model, constrained to power-law potentials only [14], can be generalized to exponential type of potentials. A detailed discussion of the main results of the paper is presented in section V, while the conclusions are given in section VI. For self-consistency and completeness, an appendix with the basic recipes of the application of the dynamical systems tools to cosmology has been added. Here we use natural units $c = 8\pi G = 1$.

II. BASIC EQUATIONS AND SET-UP

Consider the following effective action for a DBI field, ϕ [15]:

$$S_{DBI} = -\int d^4x\sqrt{|g|}\{f^{-1}(\phi)\sqrt{1+f(\phi)(\nabla\phi)^2} -f^{-1}(\phi)+V(\phi)\}, \quad (2)$$

where $f(\phi)$ is the inverse of the brane tension (also acknowledged as the warp factor of the warped throat geometry in the internal space) and, $V(\phi)$ is the potential for the DBI field, arising from quantum interactions between the D3-brane associated with ϕ, and other D-branes. In the bibliography one usually encounters given forms of the warp factor $f(\phi)$. For instance, $f(\phi) = \lambda\phi^{-4}$ (λ-constant), or, also $f(\phi) = const$. As we will immediately see, in the present paper we choose $f(\phi)$ to be related with the potential of the DBI field in the following particular form: $f(\phi) \bullet V(\phi) = 1$. The above choice leads to significant simplification of the field equations.

For a spatially flat FRW metric and arbitrary $f(\phi)$ and $V(\phi)$, the equation of motion of the DBI scalar field, ϕ, can be written in the following way:

$$\ddot{\phi} + \frac{3\partial_\phi f}{2f}\dot{\phi}^2 - \frac{\partial_\phi f}{f^2} + 3\gamma_L^{-2}H\dot{\phi} +\gamma_L^{-3}\left(\partial_\phi V + \frac{\partial_\phi f}{f^2}\right) = 0, \quad (3)$$

where $H = \dot{a}/a$ is the Hubble expansion parameter, the dot accounts for derivative in respect to the cosmic time, and the "Lorentz boost" γ_L is defined in the following way:

$$\gamma_L = \frac{1}{\sqrt{1-f(\phi)\dot{\phi}^2}}. \quad (4)$$

Alternatively the equation of motion of the DBI field can be written in the form of a continuity equation:

$$\dot{\rho}_\phi + 3H(\rho_\phi + p_\phi) = 0, \quad (5)$$

where we have defined the following energy density and pressure of the DBI scalar field:

$$\rho_\phi = \frac{\gamma_L - 1}{f} + V(\phi), \quad p_\phi = \frac{\gamma_L - 1}{\gamma_L f} - V(\phi). \quad (6)$$

Significant simplification of the above equations can be achieved after considering the following particular choice of the warp factor $f(\phi)$,

$$f(\phi) = 1/V(\phi),$$

leading to a simplified form of the effective action (2):

$$S_T = -\int d^4x\sqrt{|g|}V(\phi)\sqrt{1+(\nabla\phi)^2/V(\phi)}. \quad (7)$$

The Einstein's field equations in (flat) FRW metric, sourced by a mixture of a perfect barotropic fluid with energy density and pressure ρ_m and $p_m = \omega_m \rho_m$ respectively (ω_m is the equation of state parameter of the perfect fluid which, for dust, vanishes), and of a non-linear DBI scalar field, that can be derived from the action (7), are the following:

$$\begin{aligned} &3H^2 = \rho_m + \rho_\phi, \\ &2\dot{H} = -\left(\omega_m + 1\right)\rho_m - (\rho_\phi + p_\phi), \\ &\dot{\rho}_m = -3\left(\omega_m + 1\right) H \rho_m, \\ &\ddot{\phi} + 3\gamma^{-2} H \dot{\phi} = -\partial_\phi V (1 - 3\dot{\phi}^2/2V), \end{aligned} \tag{8}$$

where

$$\rho_\phi = \gamma V, \;\; p_\phi = -\gamma^{-1} V,$$

and the modified Lorentz boost γ is defined as:

$$\gamma = \frac{1}{\sqrt{1 - \dot{\phi}^2/V}}. \tag{9}$$

From now on we shall call the model described by equations (8,9) as "modified tachyon cosmology" (MTC).

It is remarkable that the Lagrangian densities (1) and that in the action (7) are equivalent under the change of field variables

$$\phi \to \varphi = \int \frac{d\phi}{\sqrt{V(\phi)}}. \tag{10}$$

Actually, under the transformation (10) the equations (8) above transform into the cosmological equations of standard tachyon cosmology (STC), in particular:

$$\begin{aligned} &2\dot{H} = -\left(\omega_m + 1\right)\rho_m - \frac{\dot{\varphi}^2 V(\varphi)}{\sqrt{1 - \dot{\varphi}^2}}, \\ &\frac{\ddot{\varphi}}{1 - \dot{\varphi}^2} + 3H\dot{\varphi} = -\frac{\partial_\varphi V}{V}. \end{aligned} \tag{11}$$

As we shall show in the following sections, the above change of variable will allow us to translate the results of the dynamical systems study of a DBI field with the particular choice $f(\phi) \bullet V(\phi) = 1$ (next section), to standard tachyon cosmology given by the Lagrangian density (1), thus allowing for a generalization of previous studies – see [14] for a dynamical systems study of power-law tachyon potentials – to a wider class of self-interaction potentials beyond the inverse square-law and power-law ones.

III. DYNAMICS OF A DIRAC-BORN-INFELD FIELD (CASE $f(\phi) \bullet V(\phi) = 1$)

Finding exact solutions of the cosmological equations (8) is, in general, a very difficult task. That is why we will rely on the dynamical systems tools to investigate the asymptotic structure of the DBI/tachyon cosmology. To this end we will apply the concise recipes given in the appendix (section VII). The goal will be to write the system of cosmological equations (8) in the form of an autonomous system of ordinary (ODE) as described in the appendix, so that one could associate such important dynamical systems concepts as past and future attractors (also saddle equilibrium points), with dynamical configurations – solutions – of the models. This is a powerful approach to uncover the most generic classes of solutions that are allowed by them. In order to build an autonomous system out of the system of cosmological equations (8,9) we introduce the following dimensionless phase space variables:

$$x \equiv \frac{\dot{\phi}}{\sqrt{V}}, \; y \equiv \frac{\sqrt{V}}{\sqrt{3}H} \;\Rightarrow\; \gamma = \frac{1}{\sqrt{1 - x^2}}. \tag{12}$$

After this choice of phase space variables we can write the following autonomous system of ordinary differential equations:

$$x' = (x^2 - 1)\{3x + \sqrt{3}(\partial_\phi \ln V) y\}, \tag{13}$$

$$y' = \frac{y}{2}\left\{\sqrt{3}xy(\partial_\phi \ln V) + 3\gamma_m + \right. \\ \left. + \frac{3y^2(x^2 - \gamma_m)}{\sqrt{1 - x^2}}\right\}, \tag{14}$$

where we have introduced the barotropic index for matter $\gamma_m = \omega_m + 1$ (do not confound with either γ_L or γ which play the role of the standard and modified Lorentz boosts respectively), and the tilde denotes derivative with respect to the time variable $\tau \equiv \ln a$ – properly the number of e-foldings of expansion. While deriving the ordinary differential equations (13), and (14), the following Friedmann constraint has also been considered:

$$\Omega_m \equiv \frac{\rho_m}{3H^2} = 1 - \frac{y^2}{\sqrt{1 - x^2}}. \tag{15}$$

It will be helpful to have other parameters of observational importance such as $\Omega_\phi = \rho_\phi / 3H^2$ – the scalar field dimensionless energy density parameter, and the equation of state (EOS) parameter $\omega_\phi = p_\phi / \rho_\phi$, written in terms of the variables of phase space:

$$\Omega_\phi = \frac{y^2}{\sqrt{1 - x^2}}, \;\; \omega_\phi = x^2 - 1. \tag{16}$$

Additionally, the deceleration parameter $q = -(1 + H'/H)$:

$$q = -1 + \frac{3}{2}\left[\gamma_m + \frac{y^2(x^2 - \gamma_m)}{\sqrt{1 - x^2}}\right]. \tag{17}$$

A. Exponential Potential

For an exponential self-interaction potential of the form:

$$V(\phi) = V_0 \exp(-\lambda\phi),$$

since $\partial_\phi \ln V = -\lambda = const$, then the equations (13) and (14) form a closed autonomous system of ODE:

$$x' = (x^2-1)\{3x - \sqrt{3}\lambda y\},$$
$$y' = \frac{y}{2}\left\{3\gamma_m - \sqrt{3}\lambda xy + \frac{3y^2(x^2-\gamma_m)}{\sqrt{1-x^2}}\right\}. \quad (18)$$

The phase space where to look for equilibrium points of the system of ODE (18) – corresponding to the MTC model described by (8) – can be defined as follows:

$$\Psi = \{(x,y) : -1 \le x \le 1,\ 0 \le y^4 \le 1 - x^2\}. \quad (19)$$

The equations (18) coincide with Eqs. (8,9) of [14], for a standard tachyon field with inverse square-law self-interaction potential of the form:

$$V(\phi) = V_0 \phi^{-2}.$$

Hence, the same critical points as in [14] are found in the present case, this time for the exponential potential.[2] This coincidence is easily explained through the equivalence between the present MTC model and standard tachyon cosmology (STC) model, under the change of variable (10). Actually, as it will be shown in section IV, under (10):

$$V(\phi) = V_0 e^{-\lambda\phi} \ \to\ V(\varphi) = \bar{V}_0 \left(\varphi - \varphi_0\right)^{-2}.$$

In the Figure 1 we show the trajectories in phase space for different sets of initial conditions for the model driven by an exponential potential. The free parameters have been arbitrarily set to $\gamma_m = 0.25$ and $\lambda = 1$. Due to this choice of the parameters, the equilibrium points $U = (\pm 1, 0)$ represent inflationary critical points – past attractors – in the phase space, while the matter-scaling solution is the late-time attractor. Points U are associated with ultra-relativistic behavior since $x \to \pm 1 \ \Rightarrow\ \gamma \to \infty$.

Due to our definition of the variable $x \equiv \dot{\phi}/\sqrt{V}$, the past attractor represents a scaling of the potential and of the kinetic energy of the tachyon scalar

$$x = \pm 1 \ \Rightarrow\ \dot{\phi}^2 = V(\phi).$$

[2] Recall, however, that in the above reference the variable x is defined in a simpler way: $x \equiv \dot{\phi}$.

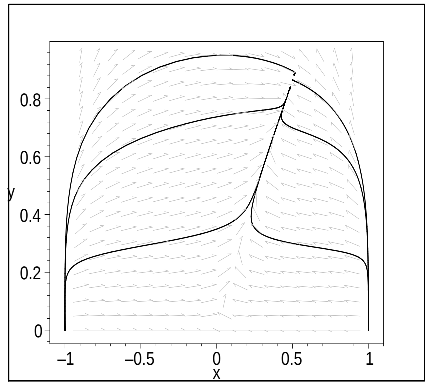

FIG. 1: Trajectories in phase space for given initial data, for the exponential potential. The free parameters have been arbitrarily set to $\gamma_m = 0.25$ and $\lambda = 1$. The matter-scaling solution is the late-time attractor, while the potential/kinetic energy scaling solution is the past attractor in the phase space.

This means that, in case the inflationary past attractor be identified with early-time inflation, it can not be associated with the slow-roll approximation which implies that the potential energy of the scalar field dominates over the kinetic one (the latter may be disregarded).

In the next section we will focus on the asymptotic properties of MTC models driven by several others self-interaction potentials of cosmological relevance.

B. Self-interaction Potentials beyond the Exponential one

As long as one considers just constant and exponential self-interaction potentials ($\partial_\phi \ln V = 0$ and $\partial_\phi \ln V = const$ respectively), the equations (13) and (14) form a closed autonomous system of ODE. However, if one wants to go further to consider a wider class of self-interaction potentials beyond the exponential one, the system of ODE (13,14) is not a closed system of equations any more, since, in general, $\partial_\phi \ln V$ is a function of the scalar field ϕ. A way out of this difficulty can be based on the method developed in [22]. In order to be able to study arbitrary self-interaction potentials one needs to consider one more variable v, that is related with the derivative of the self-interaction potential through the following expression

$$v \equiv -\partial_\phi V/V = -\partial_\phi \ln V. \quad (20)$$

Hence, an extra equation

$$v' = -\sqrt{3}xyv^2(\Gamma - 1), \quad (21)$$

has to be added to the above autonomous system of equations. The quantity $\Gamma \equiv V\partial_\phi^2 V/(\partial_\phi V)^2$ in equation (21)

TABLE I: Properties of the critical points of the autonomous system (23) for the inverse power-law potential $V(\phi) = V_0\phi^{-\lambda}$. For points M and U the variable v can take any value within the phase space, including $v = 0$.

Equilibrium Point	x	y	v	Existence	Ω_ϕ	ω_ϕ	q
M	0	0	v	Always	0	-1	$\frac{3\gamma_m-2}{2}$
dS	0	1	0	"	1	-1	-1
U	±1	0	v	"	0	0	$\frac{3\gamma_m-2}{2}$

TABLE II: Eigenvalues of the linearization matrices for the critical points in table I.

Equilibrium Point	x	y	v	λ_1	λ_2	λ_3
M	0	0	v	-3	0	$3\gamma_m/2$
dS	0	1	0	-3	0	$-3\gamma_m$
U	±1	0	v	6	0	$3\gamma_m/2$

is, in general, a function of ϕ. The idea behind the method in [22] is that Γ can be written as a function of the variable v, and, perhaps, of several constant parameters. Indeed, for a wide class of potentials the above requirement – $\Gamma = \Gamma(v)$ –, is fulfilled. Let us introduce a new function $g(v) = v^2(\Gamma(v) - 1)$ so that equation (21) can be written in the more compact form:

$$v' = -\sqrt{6}xyg(v). \tag{22}$$

Equations (13), (14), and (22) form a three-dimensional (closed) autonomous system of ODE:

$$\begin{aligned} x' &= (x^2-1)\{3x - \sqrt{3}yv\}, \\ y' &= \frac{y}{2}\{3\gamma_m - \sqrt{3}xyv + \frac{3(x^2-\gamma_m)y^2}{\sqrt{1-x^2}}\}, \\ v' &= -\sqrt{3}xyg(v), \end{aligned} \tag{23}$$

that can be safely studied with the help of the standard dynamical systems tools [23]. The function $g(v)$ can be analytically written for a wide variety of potentials. If $g(v)$ were a polynomial in v (it is the case for most quintessential potentials of cosmological interest), then, with each root $v = v_{0i}$ of the polynomial equation $g(v) = 0$ (the v_{0i}-s include the non-vanishing roots of the polynomial as well as the trivial root $v = 0$), it can be associated an equilibrium point of the autonomous system of ODE (23). When $v = 0$, $V(\phi) = V_0$, while, if $v_{0i} \neq 0$, then $V_i(\phi) \propto \exp(-v_{0i}\phi)$. Therefore, equilibrium points that arise under the requirement $g(v) = 0$, are associated either with exponential potentials in the DBI field ϕ ($v_{0i} \neq 0$), or with the constant potential ($v = 0$).

A rough inspection of the ODE (23) reveals that, independent on $g(v)$, for critical pints with $y = 0$, then, either $x = \pm1$, or $x = 0$. Therefore, independent on the functional form of the potential $V(\phi)$, the kinetic/potential energy-scaling solution ($x = \pm1 \Rightarrow \dot{\phi}^2 = V(\phi)$), and the matter-dominated solution ($x = 0$, $y = 0$, i. e., $3H^2 = \rho_m$) are always equilibrium points of the autonomous system of ODE (23). Alternatively, for equilibrium points with $x = 0$ ($y \neq 0$),

$$x' = \sqrt{3}yv, \; y' = \frac{3\gamma_m}{2}y(1-y^2) \,,$$

so that, necessarily $y = \pm1$, $v = 0$. This means that for polynomials $g(v)$, for which $v = 0$ is a root of the polynomial equation $g(v) = 0$, the de Sitter solution $3H^2 = V_0$ is an equilibrium point of the autonomous system (23).

If one started with standard tachyon cosmological equations (11) instead, then, due to the square root $\sqrt{1-\dot{\varphi}^2}$, one were forced to choose the phase space variables [14]

$$x \equiv \dot{\varphi}, \;\; y \equiv \frac{\sqrt{V}}{\sqrt{3}H}, \;\; \lambda \equiv -\frac{\partial_\varphi V}{V^{3/2}}.$$

The latter variable is necessary to close the corresponding system of ODE (compare with our variable v). Contrary to the case with the variable v, it can be shown that it is very difficult to write the parameter $\Gamma \equiv V\partial^2_\varphi V/(\partial_\varphi V)^2$ for arbitrary potentials, as a function of the variable λ. In fact, only power-law potentials, in particular $V(\varphi) \propto \varphi^{-2}$, can be analytically investigated [14]. To study the dynamics driven by other self-interaction potentials one has to rely on the obscure concept of "instantaneous critical points".

At this point one recognizes the importance of the DBI/tachyon equivalence discussed before. Actually, as already shown, the cosmological equations of standard

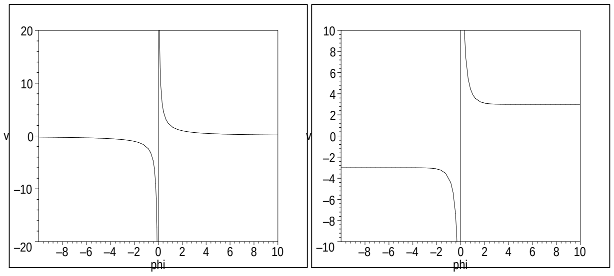

FIG. 2: A plot of the variable v vs ϕ for the potential $V = V_0\phi^{-\lambda}$ is shown for the chosen values of the free parameters: $V_0 = 1$, $\lambda = 2$ (left-hand panel), while the same plot for the potential $V = V_0 \sinh^{-\alpha}(\lambda\phi)$ is shown in the right-hand panel for $V_0 = 1$, $\lambda = 1$ and $\alpha = 3$.

tachyon cosmology (11) are mathematically equivalent to (8) under the change of variable (10). Hence, the present approach based on the variables defined in (12), (20) (in particular the variable v), enables us to further generalize former studies – with the application of the dynamical systems tools to tachyon cosmology – to a wider variety of self-interaction potentials.

A drawback of the approach undertaken here can be associated with the fact that, for potentials that vanish at the minimum – usually correlated with late-time dynamics – the variable v is undefined so that the corresponding behavior can not be properly investigated in the phase space spanned by x, y, v.

The phase space where to look for equilibrium points of the system of ODE (23), corresponding to the MTC model, can be defined as follows (we take into account only expanding universes so that only $y \geq 0$ are being considered):

$$\Psi = \{(x, y, v) : -1 \leq x \leq 1,\ 0 \leq y^4 \leq 1 - x^2,\ v\}, \quad (24)$$

where it has to be pointed out that the range of the variable v depends on the specific kind of self-interaction potential considered. Recall that, for constant and exponential self-interaction potentials one does not need to consider the latter variable, so that, the corresponding system of ODE is a two-dimensional one.

In the next subsections we study particular examples were the usefulness of the approach undertaken here is illustrated.

1. *The (inverse) power-law potential* $V(\phi) = V_0\phi^{-\lambda}$

The inverse power-law potential have been extensively studied within standard tachyon field model [14, 24]. According to the definition (20) of the variable v, for this potential one has:

$$v = \lambda\phi^{-1}, \quad (25)$$

so that the following asymptotics hold true:

$$\lim_{\phi\to\pm 0} v(\phi) = \pm\infty\ , \quad \lim_{\phi\to\pm\infty} v(\phi) = 0. \quad (26)$$

In the left hand panel of Figure 2 a plot of $v(\phi)$ vs ϕ is shown for the chosen values of free parameters: $V_0 = 1$, $\lambda = 2$. Notice that the range $\phi \in]-\infty, 0[$ is covered by negative values of the variable v, while positive values of $v > 0$ cover the range $\phi \in]0, \infty[$. In what follows, for definiteness, we will restrict ourselves to $v > 0$, so that the tachyon field variable takes values in the interval $\phi \in]0, \infty[$. In this case the function $g(v)$ in (22,23) can be written in the following way:

$$g(v) = v^2/\lambda. \quad (27)$$

The cosmic dynamics driven by this potential can be associated with a 3-dimensional phase space (24), spanned by the variables x, y, and v, where $0 \leq v < \infty$.

The equilibrium points of the autonomous system of ODE (23) in the phase space Ψ defined above, are listed in table I while the eigenvalues of the corresponding linearization matrices are shown in table II. These are non-hyperbolic critical points meaning that only limited information on their stability properties can be retrieved by means of the present linearized analysis. To help us understanding the stability properties of these points one has to rely on the phase portraits which show the structure of the phase space through phase path-probes originated by given initial data.

Existence of the matter-dominated solution (equilibrium point M in table I), is independent on the value of the variable v, meaning that this phase of the cosmic evolution may arise at early-to-intermediate times ($0 < v < \infty$), as well as at late time ($v = 0$). As seen from table II, since in this case the two non vanishing eigenvalues of the linearization matrix are of opposite sign, the matter-dominated solution is always a saddle equilibrium point of (23). This solution is inflationary whenever $\gamma_m < 2/3$.

Something similar can be said about the equilibrium point U in Tab.I. This point can be associated also either with early-time, as well as with intermediate-time dynamics, due to the fact that v can be any value. However, unlike the matter-dominated solution, the critical point U could be a past attractor (source critical point, usually associated with early-time dynamics) in the phase space, since the two non-vanishing eigenvalues of the corresponding linearization matrix are positive. Notwithstanding, since the point is a non-hyperbolic one, no final statement can be made on its stability properties until the corresponding phase portraits are drawn. The point U is also a matter-dominated solution, however,

TABLE III: Properties of the critical points of the autonomous system (23) for the potential $V = V_0[\sinh(\lambda\phi)]^{-\alpha}$. Here $v \geq \alpha\lambda$ (the constant potential $v = 0$ corresponds to the particular case where either $\alpha = 0$, or $\lambda = 0$), and we introduced the following constant parameter: $y_*^2 \equiv (\sqrt{36 - \alpha^4\lambda^4} - \alpha^2\lambda^2)/6$.

Eq. Point	x	y	v	Existence	Ω_ϕ	ω_ϕ	q
M	0	0	v	Always	0	-1	$(3\gamma_m - 2)/2$
U	±1	0	v	"	0	0	$(3\gamma_m - 2)/2$
T	$\alpha\lambda y_*/\sqrt{3}$	y_*	$\alpha\lambda$	"	1	$-y_*^2/6$	$(3\gamma_m - 2 - \alpha^2\lambda^2 y_*^2)/2$
MS	$\sqrt{\gamma_m}$	$\sqrt{3\gamma_m}/\alpha\lambda$	$\alpha\lambda$	$3\gamma_m \leq \alpha^2\lambda^2 y_*^2$	$3\gamma_m/\alpha^2\lambda^2\sqrt{1-\gamma_m}$	$-1+\gamma_m$	$(3\gamma_m - 2)/2$

TABLE IV: Eigenvalues of the linearization matrices corresponding to the equilibrium points in Tab.III. Here $\Pi \equiv \sqrt{(48\gamma_m^2\sqrt{1-\gamma_m}/\alpha^2\lambda^2) + 4 + \gamma_m(17\gamma_m - 20)}$.

Eq. Point	x	y	v	λ_1	λ_2	λ_3
M	0	0	v	-3	0	$3\gamma_m/2$
U	±1	0	v	6	0	$3\gamma_m/2$
T	$\alpha\lambda y_*/\sqrt{3}$	y_*	$\alpha\lambda$	$-2\alpha\lambda^2 y_*$	$-3 + \alpha^2\lambda^2 y_*^2/2$	$-3\gamma_m + \alpha^2\lambda^2 y_*^2$
MS	$\sqrt{\gamma_m}$	$\sqrt{3\gamma_m}/\alpha\lambda$	$\alpha\lambda$	$-6\gamma_m/\alpha$	$-3[(2-\gamma_m) + \Pi]/4$	$-3[(2-\gamma_m) - \Pi]/4$

this phase corresponds to the ultra-relativistic case (the Lorentz boost $\gamma \to \infty$) and, unlike the standard situation, it represents scaling between the kinetic and the potential energy densities of the tachyon scalar. As before, the solution is inflationary whenever $\gamma_m < 2/3$.

The late-time dynamics driven by the potential $V = V_0\phi^{-\lambda}$ is correlated with infinitely large values of the variable $\phi \to \infty$ which, in the phase space (24) ($0 \leq v < \infty$), is depicted by the equilibrium point with $v = 0$ in table I (equilibrium point dS). This point corresponds to the inflationary de Sitter solution ($3H^2 = V$). From Tab.II it is seen that this equilibrium point could be a late-time attractor since the two non-vanishing eigenvalues of the corresponding linearization matrix are both negative. However, since as already said, this is a non-hyperbolic point, only after drawing the corresponding phase portraits one is able to make conclusive statements about its stability properties.

We want to notice that the above results remain true if one considered power-law potentials with negative values of the constant parameter λ. In particular the above results can be safely extended to the quadratic potential $V(\phi) \propto \phi^2$.

In Fig.3, in order to illustrate the stability properties of the asymptotic solutions M, dS, and U, probe paths in phase space – trajectories in the phase space originated by given initial data – are drawn. As clearly seen, these trajectories emerge from the ultra-relativistic (matter-dominated) point U and converge towards the inflationary de Sitter attractor (point dS) at late times,

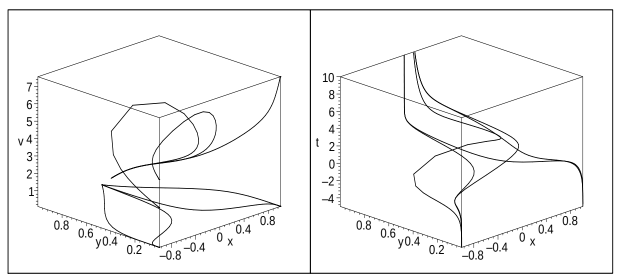

FIG. 3: In the left-hand panel trajectories in phase space for different sets initial conditions are drawn for the potential $V = V_0\phi^{-\lambda}$ ($V_0 = 1$, $\lambda = 2$, $\gamma_m = 1$ - background dust), while the flux in time of the corresponding system of ODE is depicted in the right-hand panel of the figure. Probe-paths in the phase space originate at the ultra-relativistic equilibrium point U ($x = \pm1$) while at late times approach to the de Sitter attractor dS ($x = 0,\ y = 1$).

thus confirming our suspects that U is the past attractor, while dS is the future attractor.

2. The potential $V(\phi) = V_0\,[\sinh(\lambda\phi)]^{-\alpha}$.

This potential has been formerly studied in Ref.[25] as a new cosmological tracker solution for quintessence.

According to the definition (20), for this potential one gets:

$$v = \alpha\lambda \coth(\lambda\phi), \tag{28}$$

from which it follows, in particular, that

$$\lim_{\phi\to 0} v(\phi) = \infty \ , \quad \lim_{\phi\to\pm\infty} v(\phi) = \pm\alpha\lambda. \tag{29}$$

In the right-hand panel of Fig.2 a plot of $v(\phi)$ vs ϕ is shown for the chosen values of free parameters: $V_0 = 1$, $\lambda = 1$ and $\alpha = 3$. Notice that the range of the variable $\phi \in]-\infty, 0[$ is covered by $-\infty < v \leq -\alpha\lambda$, while the range $\phi \in]0, \infty[$ is covered by $\alpha\lambda \leq v < \infty$. In what follows, for definiteness we will restrict ourselves to the interval $\alpha\lambda \leq v < \infty$. For the above potential the function $g(v)$ defined in (22,23) can be written in the following way:

$$g(v) = \frac{v^2 - \alpha^2\lambda^2}{\alpha}. \tag{30}$$

The cosmic dynamics driven by $V = V_0[\sinh(\lambda\phi)]^{-\alpha}$ can be associated with the 3-dimensional phase space (24), where the variable v is constrained to the interval $\alpha\lambda \leq v < \infty$. The equilibrium points of the autonomous system of ODE (23) in the phase space Ψ defined above, are listed in table III, while the eigenvalues of the corresponding linearization matrices are shown in Tab.IV.

As for the power-law potential, the existence of the matter-dominated solution (equilibrium point M in table III), is independent of the value of the variable v, meaning that this phase of the cosmic evolution may arise at early-to-intermediate times ($\alpha\lambda < v < \infty$), as well as at late times ($v = \alpha\lambda$). As seen from Tab.IV, since in this case the two non vanishing eigenvalues of the linearization matrix are of opposite sign, the matter-dominated solution is always a saddle equilibrium point of (23). The corresponding cosmological solution represents decelerating expansion whenever $\gamma_m > 2/3$. Unlike this, the matter-dominated equilibrium point U in Tab.III can be associated with ultra-relativistic behavior (large Lorentz boost). As already said this point represents scaling between the potential and the kinetic energies of the tachyon field. As it can be seen from the phase portraits, it is always the past attractor for any path in the phase space of the model.

Equilibrium points T (the tachyon-dominated solution) and MS (the matter-scaling solution) are associated with late-time dynamics since, according to (28), $v = \alpha\lambda$ is correlated with infinitely large values of the variable ϕ. The scalar field-dominated solution T always exits and whenever $\gamma_m > \alpha^2\lambda^2(\sqrt{36+\alpha^4\lambda^4} - \alpha^2\lambda^2)/18$ it is a stable equilibrium point (the late-time attractor), otherwise it is a saddle critical point in phase space. Whenever the matter-scaling solution MS exists,

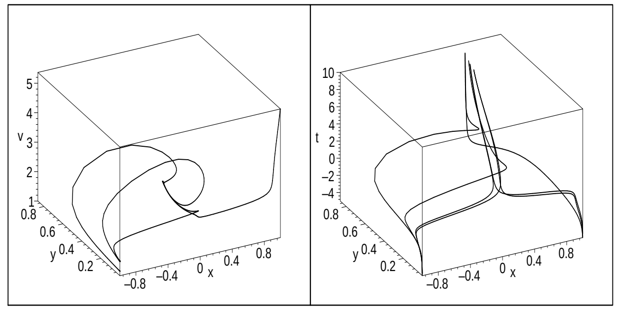

FIG. 4: Phase portrait for the model driven by the potential $V = V_0[\sinh(\lambda\phi)]^{-\alpha}$ (left-hand panel), and the flux in time of the system of ODE (23) for this case (right-hand panel). The free parameters chosen are: $\gamma_m = .2$, $\lambda = 1$ and $\alpha = 1$. It is clearly seen that the ultra-relativistic solution U is the past attractor while, due to the above choice of parameters, the matter-scaling solution MS is the late-time attractor.

it is a stable equilibrium point (the late-time attractor). This solution is always associated with accelerated expansion. As in Ref.[14], one has to take caution since the critical point MS does not exist if either $\gamma_m = 1$ (matter-dominated era), or $\gamma_m = 4/3$ (radiation domination). This is due to the fact that the existence of the matter-scaling solution requires fulfillment of the condition $0 < \gamma_m < 1$. In this sense this solution can not be associated with a realistic model of dark energy.

In the figure 4 the phase portrait for this case is depicted. The above discussed behavior is clearly illustrated by the figure. The free parameters were taken in such a way that the MS solution exists ($\gamma_m = .2$, $\lambda = 1$, $\alpha = 1$). In correspondence, the ultra-relativistic phase U is the past attractor, while the matter-scaling solution MS is the late-time (inflationary) attractor.

IV. DBI/TACHYON EQUIVALENCE

As already shown, there is a mathematical equivalence between the DBI model given by the Lagrangian density (7), and the tachyon model portrayed by the Lagrangian (1), under the transformation (10):

$$\phi \to \varphi = \int \frac{d\phi}{\sqrt{V(\phi)}}.$$

Equivalence under (10) implies a residual equivalence between magnitudes of physical relevance, in particular:

$$\Omega_\phi = \Omega_\varphi, \ \ \omega_\phi = \omega_\varphi, \ \ q(\phi) = q(\varphi) \equiv -(1 + \frac{\dot{H}}{H^2}). \tag{31}$$

It is evident that, once the functional form of the self-interaction potential $V(\phi)$ (or $V(\varphi)$) is known, the func-

tional relationship $\varphi=\varphi(\phi)$ (or $\phi=\phi(\varphi)$) can be obtained through integration in quadratures, so that one is able to transform the potential $V(\phi)=V(\phi(\varphi))\;\rightarrow\; V=V(\varphi)$ (or $V(\varphi)=V(\varphi(\phi))\;\rightarrow\; V=V(\phi)$). Actually, from (10) it follows that

$$\varphi-\varphi_0=\int\frac{d\phi}{\sqrt{V(\phi)}},\text{ or, }\phi-\phi_0=\int\sqrt{V(\varphi)}d\varphi.\quad(32)$$

By using equation (32) it can be found that the transformation (10) implies the following transformations between modified tachyon potentials $V(\phi)$ and standard tachyon ones $V(\varphi)$:

$$V(\phi)=V_0e^{-\lambda\phi}\;\rightarrow\;V(\varphi)=\bar{V}_0(\varphi-\varphi_0)^{-2},\qquad(33)$$

where $\bar{V}_0\equiv 4/\lambda^2$, and φ_0 is an integration constant. For the inverse power-law potential one gets that

$$V(\phi)=V_0\phi^{-2\lambda}\;\rightarrow\;V(\varphi)=\bar{V}_0(\varphi-\varphi_0)^{-2n},\qquad(34)$$

where $\bar{V}_0=[V_0/(\lambda+1)^{2\lambda}]^{1/\lambda+1}$, and $n=\lambda/(\lambda+1)$. Additionally, for the sinh-like potential $\propto\sinh^{-2}(\lambda\phi)$, one obtains the following equivalence

$$V(\phi)=V_0\sinh^{-2}(\lambda\phi)\;\rightarrow\;V(\varphi)=\frac{\bar{V}_0}{\varphi^2-\varphi_0^2},\qquad(35)$$

where $\bar{V}_0=1/\lambda^2$ and $\varphi_0^2=1/V_0\lambda^2$.

Notice that under the transformation (10) the DBI exponential potential is equivalent to the square-law tachyon potential $\propto\varphi^{-2}$ studied in [14], while the (inverse) power-law DBI potential $\propto\phi^{-2\lambda}$ is equivalent to a (inverse) power-law tachyon potential $\propto\varphi^{-2n}$, not fully investigated in the same reference. I. e., the inverse power-law potential is not transformed under (10). In the later case, the only difference of physical significance is in the power of the potential since $\lambda\rightarrow n=\lambda/(\lambda+1)$. It is seen that, for positive $\lambda>0\;\Rightarrow\;n\leq 1$.

The above discussed DBI/tachyon equivalence opens up the possibility to apply the present approach to investigate the dynamics of standard tachyon cosmology for self-interaction potentials beyond the square-law potential which has been studied in detail in Ref.[14]. Actually, consider, for instance, the exponential self-interaction potential for the standard tachyon:

$$V(\varphi)=V_0e^{\mu\varphi}.$$

By using the relationship (32) it can be shown that:

$$V(\varphi)=V_0e^{\mu\varphi}\;\rightarrow\;V(\phi)=\frac{\mu^2}{4}\phi^2.$$

Fortunately this case for the DBI-modified dynamics has been already studied in subsection III B 1 (just replace $\lambda\rightarrow -2$ in equations (25) and (27), and bear in mind that the non-negative range of the variable v is associated with negative values of the tachyon scalar $\phi\in]-\infty,0[$). It remains just to translate the corresponding results so that one could discuss their physical implications for the tachyon cosmological dynamics.

V. DISCUSSION

Thanks to the formal mathematical equivalence under the transformation (10), among standard tachyon dynamics depicted by Sen's Lagrangian (1) and the dynamics of a DBI field given by the Lagrangian within action (7), the approach undertaken in this paper enables applying the standard tools of the dynamical systems to investigate the cosmic dynamics driven by a wide variety of (scalar) tachyon self-interaction potentials, without resorting to such obscure concepts as "instantaneous critical points", whose physical relevance is suspicious. Actually, if such a mathematical (and dynamical) equivalence is taken into consideration, the results obtained in section III – after applying the linear analysis to study the dynamics of the model of (7) – can be safely translated to the case of the standard tachyon model portrayed by the Lagrangian density (1).

As shown in the former section there is a full equivalence between inverse power-law DBI potential $\propto\phi^{-2\lambda}$ and that of the tachyon $\propto\varphi^{-2n}$, so that, for this kind of potential the results displayed in the tables I and II for the DBI field hold true for the standard tachyon cosmological model of [14], which means in turn, that a detailed study of the dynamics driven by this tachyon potential is indeed possible. According to the results of section III (see Tabs. I and II), for the potential $V(\varphi)\propto\varphi^{-2n}$, whenever $0<n\leq 1$, one obtains that the de Sitter solution – point dS in Tab. I – is always the late-time attractor in the phase space, while the ultra-relativistic matter-dominated solution – point U in Tab. II – is the past attractor from which the phase paths originate. The matter-dominated solution M is always a saddle in the phase space. Therefore, the standard tachyon cosmology model driven by the inverse power-law potential could be a nice scenario to address the late-time cosmic acceleration.

The study of the asymptotic properties of the tachyon model for the exponential potential is mathematically equivalent to the study of the asymptotic properties of the DBI cosmological model for the quadratic potential $V(\phi)\propto\phi^2$, which is a particular case of the study presented in III B 1 when the constant parameter λ is replaced by the particular negative value -2. The only think to be changed is the phase space itself if one keeps $\phi\in]0,\infty[$ — in this case, in place of the half of the phase space (24) corresponding to positive v-s, one has to consider the complementary half defined by negative v-s instead —, or one might keep intact the phase space at the cost that the tachyon field itself takes values in the interval $\phi\in]-\infty,0[$.

A remarkable property of the tachyon model studied here is that, independent of the particular functional form of the self-interaction potential $V(\phi)$ considered, the matter-dominated solutions M and U – the ultra-relativistic matter-dominated solution, are always equilibrium points of the corresponding autonomous system of ODE (23) (see tables I,III). A straightforward inspection of the equations (23) reveals why this happens. Actually, a crude inspection of the equations in the system of ODE (23) shows that, independent of the functional form of the function $g(v)$ and of the value of the variable v, since for $y = 0$ the system (23) reduces to the simplified system of equations:

$$x' = 3x(x^2 - 1), \quad y' = 0, \quad v' = 0,$$

then, for $x = 0$ and $x = \pm 1$, the corresponding points (x, y, v) in phase space: $M = (0, 0, v)$, and $U = (\pm 1, 0, v)$, both are equilibrium points of the system of ODE (23). Since the existence of these points is independent of the value of the variable v, both phases of the cosmic evolution may arise at early, intermediate, as well as at late times. In fact, the point M is always a saddle critical point, while U is the past attractor for any path in the phase space of the model, otherwise, U is the point in phase space from which all of the phase trajectories are repelled.

From the analysis of the equations (23) it also arises that, in general, for potentials for which the function $g(v)$ is a polynomial in v (it happens for most part of known quintessential potentials), and the polynomial equation $g(v) = 0$ has non-vanishing roots $v = v_{0i} \neq 0$, since in this case the system (23) reduces to the autonomous system of ODE (18) for an exponential potential $V \propto e^{\lambda\phi}$ ($\lambda = v_0$), the late-time dynamics of the tachyon field can be either the scalar field-dominated solution, or the matter-scaling phase. This conclusion is quite robust and has been formerly stated in [22] in a different context.

We can use the formal DBI/tachyon mathematical equivalence to establish links between previous results within the DBI bibliography and the study of equivalent tachyon field cases. In reference [21], for instance, the authors study (among other cases) the late-time dynamics of a DBI model based on the action (2), with exponential potential and brane tension (here we use the notation of [21]):

$$V(\phi) = \sigma e^{-\lambda\phi}, \quad f(\phi) = \gamma e^{-\mu\phi} .$$

In the particular case where $\lambda + \mu = 0 \;\Rightarrow\; f \bullet V = \gamma\sigma$, a new class of solutions arise, for which $0 < \tilde{\gamma} < 1$ ($\tilde{\gamma} = \gamma^{-1} = \sqrt{1 - x^2}$), i. e., $x \neq 0$, and $x \neq \pm 1$. These solutions are the matter-scaling solution and the scalar field-dominated (kinetic/potential energy-scaling) phase. It is interesting noting that, under the field replacement (10), and $f \bullet V = 1$, the exponential potential above transforms into the (inverse) square tachyon potential

$$V(\varphi) = \frac{4/\lambda^2}{(\varphi - \varphi_0)^2} ,$$

so that, given that $\gamma\sigma = 1$, the former results are equivalent to the results of the dynamical systems study of tachyon cosmology with an inverse square potential of reference [14]. Actually, the equilibrium points (c) and (d1,d2) in Tab. I of Ref. [14] are the above mentioned matter-scaling, and the scalar (tachyon) field-dominated solutions respectively. As long as we know, no such equivalence between DBI and tachyon dynamics has been reported before.

VI. CONCLUSIONS

In the quest for alternative models of inflation (here we include both primordial early-time and late-time inflation), the so called tachyon models and their generalization: the scalar DBI field models, have played an important role. Due to complexity of the analysis, in particular in connection with difficulties to obtain closed (autonomous) systems of ordinary differential equations out of the corresponding cosmological equations, the study of the tachyon dynamics has been performed for a limited number of particular tachyon potentials.

In the present paper we have proposed an approach based on an existing mathematical equivalence between the dynamics of a DBI field ϕ with the following relationship between the warp factor f of the internal geometry, and the DBI potential V: $f \bullet V = 1$, and the dynamics of a standard tachyon field φ driven by the Lagrangian

$$\mathcal{L} = -V\sqrt{1 + (\nabla\varphi)^2} ,$$

under the field redefinition $\varphi = \int d\phi/\sqrt{V(\phi)}$. The above equivalence allows to translate the results of dynamical systems studies of DBI models with the condition $f \bullet V = 1$, for given DBI potentials, to the corresponding studies of the equivalent tachyon models. The discussed equivalence, in conjunction with the application of an approach developed in Ref. [22], which allows to apply the standard dynamical systems tools to the study of a wide variety of self-interaction potentials, provided that the function

$$g(v) \equiv v^2(\Gamma - 1) ,$$

where

$$v \equiv -\frac{\partial_\phi V}{V}, \quad \Gamma = \frac{V\partial_\phi^2 V}{(\partial_\phi V)^2} ,$$

can be written as a polynomial in the new phase space variable v, makes possible to investigate tachyon models driven by a broad class of potentials. The usefulness of our combined approach has been illustrated with the detailed study of the following tachyon potentials

$$V(\varphi) = V_0\, \varphi^{-2n}, \quad V(\varphi) = \frac{V_0}{\varphi^2 - \varphi_0^2} ,$$

where the latter potential is just a particular case of the former one for $n = 1$, $\varphi_0 = 0$. It has been demonstrated

that the φ^{-2n} potential is equivalent to the inverse power-law DBI potential $\phi^{-2\lambda}$ ($n = \lambda/(\lambda+1)$), while the inverse square tachyon potential φ^{-2} is equivalent to the exponential DBI potential $\exp -\lambda\phi$. The particular inverse square tachyon potential $(\varphi^2 - \varphi_0^2)^{-1}$ is equivalent to the sinh-like potential $\sinh^{-2}(\lambda\phi)$, therefore, different kinds of DBI potentials might be equivalent to the same tachyon potential. In spite of the fact that only the dynamics driven by the inverse power-law tachyon potential φ^{-2n} has been studied in details, nonetheless, the combined method used here can be applied to other kinds of potentials, provided that the function $g(v)$ can be written as a polynomial on v for the equivalent DBI potential $V(\phi)$.

Amongst the most interesting results of the present study we can list the following:

- A formal mathematical equivalence between DBI models with the following particular relationship: $f \bullet V = 1$, between the warp factor f and the DBI potential V, and tachyon cosmological models originated from the Lagrangian (1), under the field redefinition (10), has been found. Under the above DBI/tachyon mathematical equivalence, results of different, otherwise unrelated investigations, are fully equivalent, so that, for instance, several results of studies of DBI models with exponential brane tension and potential [21], and the results of the dynamical systems studies of tachyon cosmological models with inverse square potential [14], are revealed to be equivalent.

- It is revealed that independent of the functional form of the potential, the matter-dominated solution and the ultra-relativistic (also matter-dominated) solution, are always associated with equilibrium points in the phase space of the tachyon models. The latter is always the past attractor, while the former is a saddle critical point.

- It has been demonstrated that, in general, for DBI potentials for which the function $g(v)$ can be written as a polynomial in v (it is the case for most quintessential potentials of cosmological interest):

  $$g(v) = \sum_n (v - v_0)^n ,$$

  the late-time dynamics is associated with either the de Sitter solution if at the critical point $v = 0$, is a root of the polynomial equation $g(v) = 0$, or with matter-scaling and/or scalar field-dominated solutions if at the critical point, the given root $v = v_0 \neq 0$.

The present approach can be safely (and moderately easy) applied to consider new classes of DBI potentials that are equivalent to other classes of tachyon potentials beyond the (inverse) power-law one.

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VII. APPENDIX: DYNAMICAL SYSTEMS

Here we include brief tips of how to apply the dynamical systems tools to situations of cosmological interest. In order to apply these tools one has to follow the steps enumerated below:

1. To identify the phase space variables that allow writing the system of cosmological equations in the form of an autonomus system of ordinary differential equations (ODE), say:[3]

   $$x_i = (x_1, x_2, ...x_n) .$$

2. With the help of the chosen phase space variables, to build an autonomous system of ODE out of the original system of cosmological equations (τ is the time-ordering variable, not necessarily the cosmic time):

   $$\frac{dx_i}{d\tau} = f_i(x_1, x_2, ...x_n) .$$

   Notice that the RHS of these equations do not depend explicitly on τ (that is the reason why the system is called autonomous).

3. To identify the phase space spanned by the chosen variables $(x_1, x_2, ...x_n)$, that is relevant to the cosmological model under study. This amounts, basically, to define the range of the phase space variables that is appropriate to the problem at hand:

   $$\Psi = \{(x_1, x_2, ...x_n) : \text{bounds on the } x_i\text{-s}\} .$$

4. Finding the equilibrium points of the autonomous system of ODE, amounts to solve the following system of algebraic equations on $(x_1, x_2, ...x_n)$:

   $$f_i(x_1, x_2, ...x_n) = 0 .$$

5. Next one linearly expands the equations of the autonomous system of ODE in the neighborhood of the equilibrium points $\bar{p}_k = p_k(\bar{x}_1, \bar{x}_2, ...\bar{x}_n)$, $k =$

[3] There can be several different possible choices, however, not all of them allow for the minimum possible dimensionality of the phase space.

$1, 2, ...m$:[4] I. e., one replaces $x_i \to \bar{x}_i + e_i$, where e_i are the small (linear) perturbations around the equilibrium points. Hence the system of ODE becomes a system of linear equations to determine the evolution of the e_i-s:

$$\frac{de_i}{d\tau} = \bar{f}_i + \sum_{j=1}^{n} \left(\frac{\partial f_i}{\partial x_j} \right)_{\bar{p}} e_j + \mathcal{O}(e_i^2) \ ,$$

otherwise, since $\bar{f}_i = f_i(\bar{p}) = 0$, then

$$\frac{de_i}{d\tau} = \sum_{j}^{n} [M(\bar{p})_i^j] \ e_j + \mathcal{O}(e_i^2) \ ,$$

where we have introduced the linearization or Jacobian matrix $[M_i^j] = \partial f_i / \partial x_j$.

6. The next step is to solve the secular equation to determine the eigenvalues λ_i of the linearization matrix at the given equilibrium point $\bar{p}$:

$$\det |M(\bar{p})_i^j - \lambda \ U_i^j| = 0 \ ,$$

where $[U_i^j]$ is the unit matrix.

7. Once the eigenvalues of the linearization around a given equilibrium point $\bar{p}$ have been computed, the evolution of the perturbations is given by

$$e_i(\tau) = \sum_{j}^{n} (e_0)_i^j \exp{(\lambda_j \tau)} \ ,$$

where the amplitudes $(e_0)_i^j$ are constants of integration.

[4] In general the number of equilibrium points is different from the dimension of the phase space: $m \neq n$.

If all of the eigenvalues have negative real parts, the perturbations decay with τ, i. e., the equilibrium point is stable against linear perturabtions. The corresponding equilibrium point is said to be a future attractor. If at least one of the eigenvalues has positive real part, the perturbations grow with τ so that these are not stable in the direction spanned by the given eigenvalue. Hence the point is said to be a saddle. The perturbations around a given equilibrium point are unstable, in other words the point is a past attractor (a source point in the phase space), if all of the eigenvalues have positive real parts. Points whose linearization is characterized by complex eigenvalues are said to be spiral equilibrium points, and are commonly associated with oscillatory behavior of the corresponding solution. If at least one of the eigenvalues has a vanishing real part, the equilibrium point is said to be non-hyperbolic. In the latter case, in general, and unless some of the non-vanishing real parts of the eigenvalues are of opposite sign, one can not give conclusive arguments on the stability of the equilibrium point. Other techniques have to be applied.

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